A1 Selected problems on Setting up Algebraic Expressions

Marieke Bor, Françoise Chenevotot, Paul Drijvers, Brigitte Grugeon-Allys, version 14-1-2019, Advise-Me

This set of tasks concerns the domain of Relationship. In particular, it focuses on setting up algebraic expressions and simplifying them, and on setting up equations and inequalities and solving them. As such, it will form the point of departure for the design of automated diagnostics with intermediate steps, core in the Advise-Me project. Criteria for tasks to be included in this set are:

- The task concerns the domain Relationships and should fit in the 12-15 year old student's competence, as identified in the domain model in A2 (see final page of this document);
- The task involves setting up algebraic expressions, equations, and inequalities, as well as simplifying and solving them;
- The task involves multi-step solutions.

The student model with all its nodes is listed below. With the number of appearances in brackets.

- R1: Set up expressions, equations and inequalities
    - R11: Set up numerical expressions (24)
    - R12: Set up algebraic equations (5)
        - R121: Set up algebraic expressions (8)
            - R1211: Choose variables (7)
            - R1212: Set up geometric model (14)
            - R1213: Set up linear model (10)
            - R1214: Set up quadratic model (8)
            - R1215: Set up exponential model (1)
    - R13: Set up algebraic inequalities (3)

- R3: Simplify algebraic expressions and solve equations
    - R31: Use numerical expressions (31)
    - R32: Solve equations (0)
        - R321: Substitute (4)
        - R322: Simplify expressions (0)
            - R3221: Expand expressions (8)
            - R3222: Divide out common factors (1)
            - R3223: Combine like terms (5)
        - R323: Simplify by addition and subtraction (7)
        - R324: Simplify by multiplication and division (7)

For each of the tasks possible solution steps are coded.

For example: Ans2Strat1Step3.

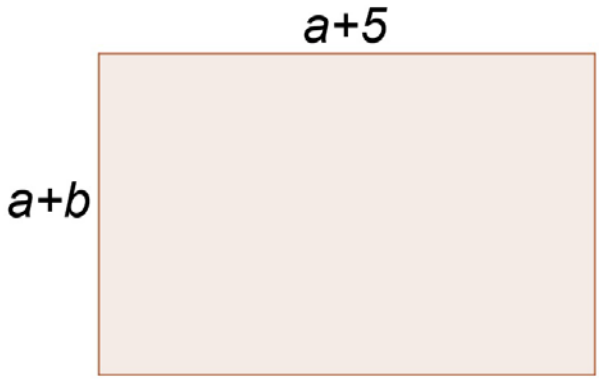
- ‘Ans2’ means this step is part of question 2. If a task has only one question, the code starts with Ans1.
- ‘Strat1’ denotes the step is part of strategy 1.
- ‘Step3’ gives the number of the step.

Most solution steps have a connection with nodes of the user model. These nodes are listed with the solution steps.

1. Rectangle Area (R1, R2 & R3)

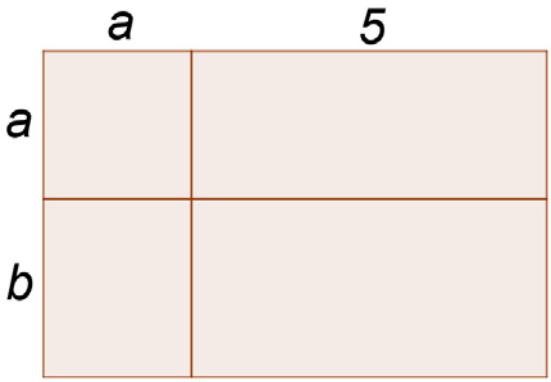
Task (source Cito inspired by Pépite)

Here is a rectangle:



Q1: Find an expression for the area of the rectangle, using a and b.

<multi-line formula editing window or open answer box>



Q2: Find expressions for the area of each of the four partial rectangles, using a and b, and add the four to find an expression for the total area.

< multi -line formula editing or open answer box >

Q3: Check the equivalence of the two expressions you found.

< multi -line formula editing window or open answer box >

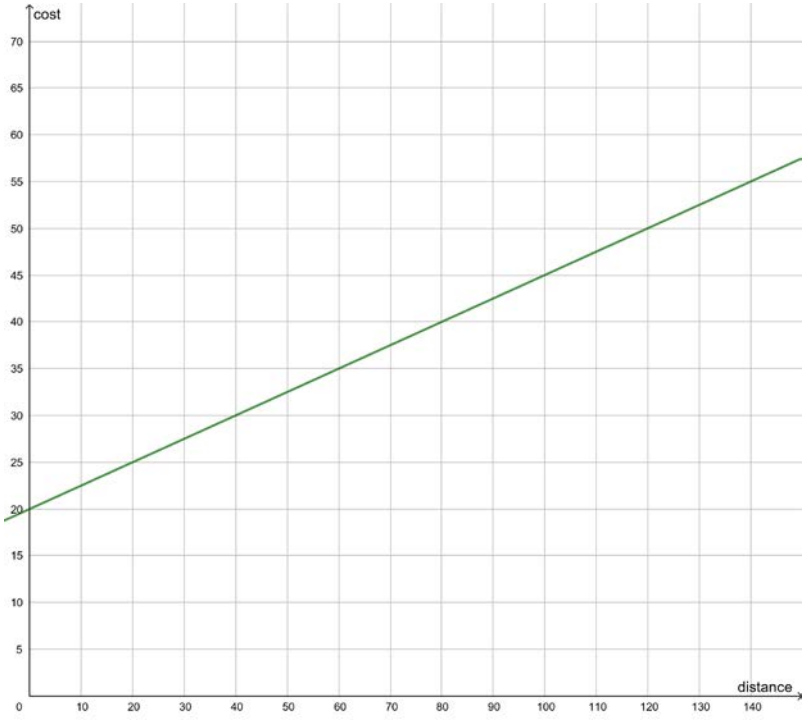
Possible steps:

Code	Solution step	Node user model
Ans1Strat1Step1	(a+5)*(a+b)	R1212: Set up geometric model R1214: Set up quadratic model
Ans2Strat1Step11	a^2	R1212: Set up geometric model R1214: Set up quadratic model

Ans2Strat1Step12	5*a	R1212: Set up geometric model R1214: Set up quadratic model
Ans2Strat1Step13	a*b	R1212: Set up geometric model R1214: Set up quadratic model
Ans2Strat1Step14	5*b	R1212: Set up geometric model R1214: Set up quadratic model
Ans2Strat1Step2	a^2 + 5a + ab + 5b	
Ans3Strat1Step1	(a+5)*(a+b) = a^2 + 5a + ab + 5b	R12: Set up algebraic equations R3221: Expand expressions

2. Car rental (R1, R2 & R3)

Task (source CITO)



Mark wants to rent a car for a day. The above graph shows the total costs depending on the number of kilometers he drives, if he hires at the "Rent-it" company: you pay € 20.00 for the day plus € 0.25 for every kilometer you drive. Company "Go for it" offers a day price of € 43.00 without extra charge per km.

Calculate how many km Mark needs to drive at least as to prefer the "Go for it" offer and explain your answer. Write down your intermediate steps.

< multi-line formula editing window or open answer window>

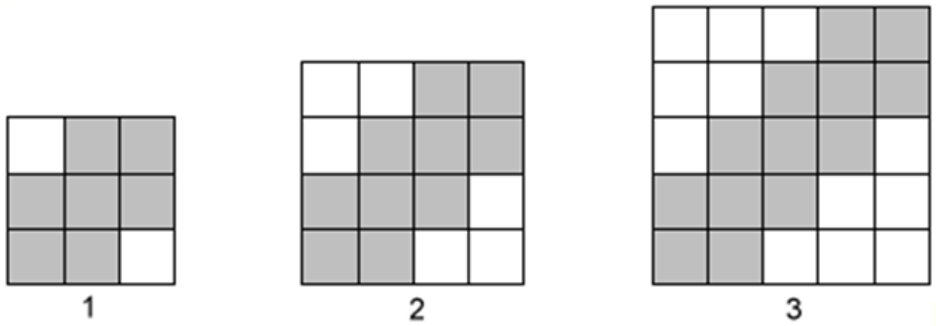
Possible steps:

Code	Solution step	Node user model
Ans1Strat1Step1	Choose variable: dist (distance) Choose variable: cost	R1211: Choose variables
Ans1Strat1Step2	cost = 20.00 + 0.25*dist	R1213: Set up linear model
Ans1Strat1Step31	20.00 + 0.25*dist = 43.00	R12: Set up algebraic equations
Ans1Strat1Step41	0.25*dist = 43.00 - 20.00 = 23.00	R323: Simplify by addition and subtraction
Ans1Strat1Step51	dist = 23.00 / 0.25 = 92.00	R324: Simplify by multiplication and division
Ans1Strat1Step61	Answer: 92 km or more	
Ans1Strat1Step32	20.00 + 0.25*dist > 43.00	R13: Set up algebraic inequalities
Ans1Strat1Step42	0.25*dist > 23.00	R323: Simplify by addition and subtraction

Ans1Strat1Step52	dist > 92.00	R324: Simplify by multiplication and division
Ans1Strat1Step62	Answer: 92 km or more	
Ans1Strat2Step1	20.00 + 0.25 * <number > = <other number>	R11: Set up numerical expressions R31: Use numerical expressions
Ans1Strat2Step2	20.00 + 0.25 * 92 = 43.00	R11: Set up numerical expressions R31: Use numerical expressions
Ans1Strat2Step3	Answer: 92 km or more	

3. Pattern (R1, R2 & R3)

Task (source CITO)



The square tile patterns above consist of white and grey tiles.
Pattern 1 contains 7 grey tiles.
Pattern 2 contains 10 grey tiles.
And so on: the pattern can be extended infinitely.

Which pattern is the first to have more than 50 grey tiles?

<multi-line formula editing window or open answer box >

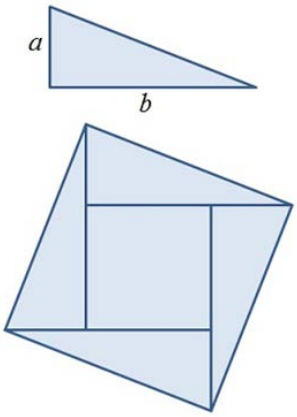
Possible steps:

Code	Solution step	Node user model
Ans1Strat1Step11	Choose variables: n (pattern number) Choose variable: G (number of grey tiles)	R1211: Choose variables
Ans1Strat1Step12	Recognize starting number is 4 Each step 3 are added	
Ans1Strat1Step2	G = 4 + 3*n	R1213: Set up linear model
Ans1strat1Step31	4 + 3*n = 50	R12: Set up algebraic equations
Ans1Strat1Step41	3*n = 50 - 4 = 46	R323: Simplify by addition and subtraction
Ans1Strat1Step51	n = 46 / 3 = 15.33	R324: Simplify by multiplication and division
Ans1Strat1Step61	Answer: pattern 16	
Ans1strat1Step32	4 + 3*n > 50	R13: Set up algebraic inequalities
Ans1strat1Step42	3*n > 46	R323: Simplify by addition and subtraction
Ans1strat1Step52	n > 15.33	R324: Simplify by multiplication and division
Ans1strat1Step62	Answer: pattern 16	
Ans1Strat2Step1	Calculate a value of the first patterns: 7, 10, 13, 16, …	R31: Use numerical expressions
Ans1Strat2Step2	Calculate pattern 16 has 52 grey tiles	R31: Use numerical expressions
Ans1Strat2Step3	Answer: pattern 16	

Ans1Strat3Step1	Recognize each step 3 grey tiles are added	
Ans1Strat3Step2	After pattern 1, add 50-7=43 grey tiles	R31: Use numerical expressions
Ans1Strat3Step3	add 43/3 = 14.33 steps	R31: Use numerical expressions
Ans1Strat3Step4	14.33 + 1 = 15.33	R31: Use numerical expressions
Ans1Strat3Step5	Answer: pattern 16	

4. Making a square (R1 & R2)

Task (source CITO)
On your right hand side you see a right-angled triangle with adjacent sides of length a and b. Four of these triangles are put together in such way that they form a big square that includes a smaller square.



Express and reduce the area A of the big square in a and b. Write down your intermediate steps.

<multi-line formula editing window or open answer window>

Possible steps:

Code	Solution step	Node user model
Ans1Strat1Step1	Set up algebraic expression of area triangle = a*b/2	R1212: Set up geometric model R1214: Set up quadratic model
Ans1Strat1Step2	Set up algebraic expression of side small square = b – a	R1212: Set up geometric model
Ans1Strat1Step3	Set up algebraic expression of area small square = (b - a)^2	R1212: Set up geometric model R1214: Set up quadratic model
Ans1Strat1Step4	Set up algebraic expression of area big square: A = 4*a*b/2 + (b - a)^2	R1212: Set up geometric model
Ans1Strat1Step5	A = 2*a*b + b^2 - 2*a*b + a^2	R3221: Expand expressions
Ans1Strat1Step6	Answer: A = b^2 + a^2	R3223: Combine like terms
Ans1Strat2Step1	Set up algebraic expression of length hypotenuse = sqrt(a^2 + b^2)	R1212: Set up geometric model
Ans1Strat2Step2	Set up algebraic expression of area big square: A = (sqrt(a^2 + b^2))^2	R1212: Set up geometric model R1214: Set up quadratic model
Ans1Strat2Step3	Answer: A = a^2 + b^2	

5. Magical trick? (R1, R2 & R3)

Task (source Pépite)
A student says to her peer: “Choose a number, add 8, multiply the result by 3, subtract 4, add the initial number, divide by 4, add 2, and subtract the initial number. You will end up with 7.” Is this true for any starting number? Explain your answer by writing down intermediate steps.

< multi-line formula editing window or open answer window>

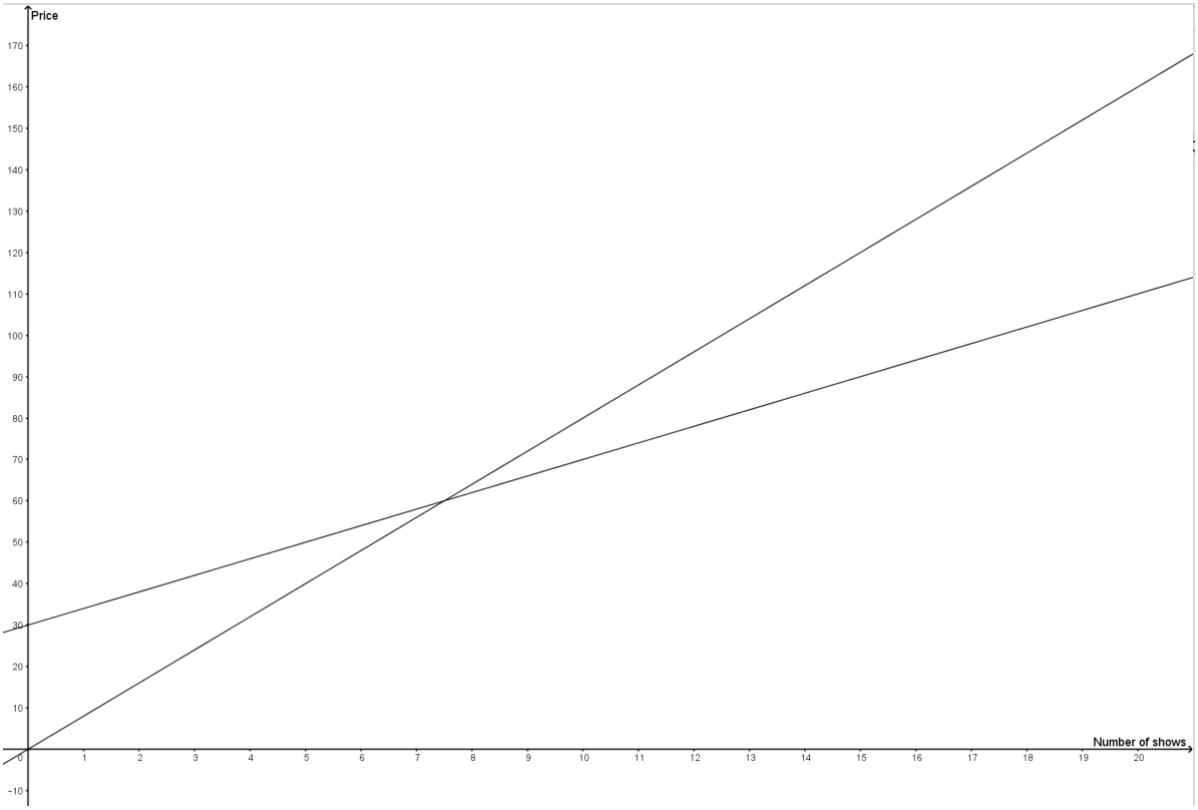
Possible steps:

Code	Solution step	Node user model
Ans1Strat1Step1	Choose variable: x	R1211: Choose variables
Ans1Strat1Step2	((x + 8)*3 - 4 + x)/4 + 2 – x	R121: Set up algebraic expressions
Ans1Strat1Step3	(3x + 24 – 4 + x)/4 + 2 – x	R3221: Expand expressions
Ans1Strat1Step4	(4x + 20)/4 + 2 – x	R3223: Combine like terms
Ans1Strat1Step5	x + 5 + 2 – x	R3221: Expand expressions
Ans1Strat1Step6	7	R3223: Combine like terms
Ans1Strat2Step1	x+8	R121: Set up algebraic expressions R1211: Choose variables
Ans1Strat2Step2	(x + 8)*3 = 3x + 24	R121: Set up algebraic expressions R3221: Expand expressions
Ans1Strat2Step3	3x + 24 – 4 = 3x + 20	R121: Set up algebraic expressions R31: Use numerical expressions
Ans1Strat2Step4	3x + 20 + x = 4x + 20	R121: Set up algebraic expressions R3223: Combine like terms
Ans1Strat2Step5	(4x+20)/4 = x + 5	R121: Set up algebraic expressions R3221: Expand expressions
Ans1Strat2Step6	x + 5 + 2 = x + 7	R121: Set up algebraic expressions R31: Use numerical expressions
Ans1Strat2Step7	x + 7 – x = 7	R121: Set up algebraic expressions R3223: Combine like terms
Ans1Strat3Step1	((<number> + 8)*3 - 4 + <number>)/4 + 2 – <number>	R11: Set up numerical expressions
Ans1Strat3Step2	((<number> + 8)*3 - 4 + <number>)/4 + 2 – <number> = 7	R31: Use numerical expressions
Ans1Strat4Step1	<number> + 8 = <answer1>	R11: Set up numerical expressions R31: Use numerical expressions
Ans1Strat4Step2	<answer1> * 3 = <answer2>	R11: Set up numerical expressions R31: Use numerical expressions
Ans1Strat4Step3	<answer2> - 4 = <answer3>	R11: Set up numerical expressions R31: Use numerical expressions

Ans1Strat4Step4	<answer3> + <number> = <answer4>	R11: Set up numerical expressions R31: Use numerical expressions
Ans1Strat4Step5	<answer4> / 4 = <answer5>	R11: Set up numerical expressions R31: Use numerical expressions
Ans1Strat4Step6	<answer5> + 2 = <answer6>	R11: Set up numerical expressions R31: Use numerical expressions
Ans1Strat4Step7	<answer6> - <number> = 7	R11: Set up numerical expressions R31: Use numerical expressions

6. Theatre rate (R1, R2 & R3)

Task (source Paris)



A theater programs 20 shows during the year and offers two rates:

R1: a subscription of 30 euros and an entrance fee of 4 euros per show

R2: no subscription and an entrance fee of 8 euros per show

For each rate, the graph shows the total cost in euros depending on the number of shows.

Q1 Decide which of the graphs represents R1, and which one R2.

<Answer box:> R1 o red R2: o red

o blue o blue

Q2 Set up an equation or inequality to find out from which number of shows the R1 offer is cheaper than the R2 offer. Write down your intermediate steps.

<multi-line formula editing window or open answer box>

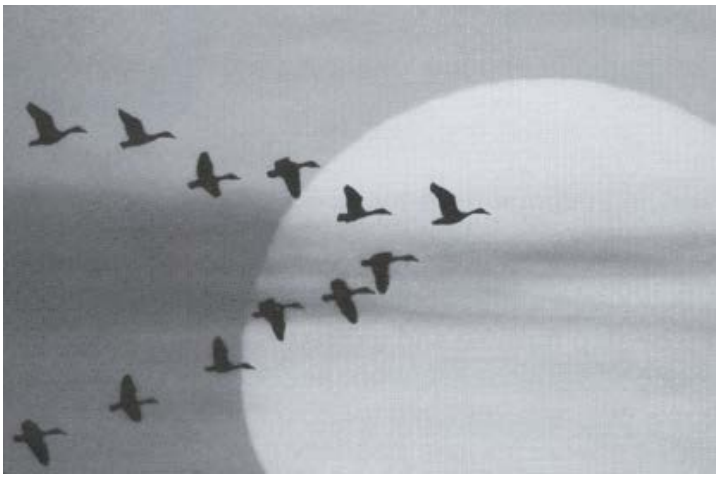
Possible steps:

Code	Solution step	Node user model

Ans2Strat1Step1	Choose variable: n (number of shows) Choose variable: p (price)	R1211: Choose variables
Ans2Strat1Step21	R1: p = 30 + 4*n	R1213: Set up linear model
Ans2Strat1Step22	R2: p = 8*n	R1213: Set up linear model
Ans2Strat1Step31	30 + 4*n = 8*n	R12: Set up algebraic equations
Ans2Strat1Step41	4*n = 30	R323: Simplify by addition and subtraction
Ans2Strat1Step51	n = 30 / 4 = 7.5	R324: Simplify by multiplication and division
Ans2Strat1Step61	Answer: 8 shows	
Ans2Strat1Step32	30 + 4*n < 8*n	R13: Set up algebraic inequalities
Ans2Strat1Step42	4*n > 30	R323: Simplify by addition and subtraction
Ans2Strat1Step52	n > 7.5	R324: Simplify by multiplication and division
Ans2Strat1Step62	Answer: 8 shows	
Ans2Strat2Step1	Try <number> 30 + 4 * <number> ≠ 8 * <number>	R11: Set up numerical expressions R31: Use numerical expressions
Ans2Strat2Step2	Try <8> 30 + 4 * 8 < 8 * 8	R11: Set up numerical expressions R31: Use numerical expressions
Ans2Strat2Step3	Answer: 8 shows	

7. V-patterns (R1 & R2)

Task (source Kindt, 2003, Oefeningen in algebra)



In groups, birds may fly in a V-shaped formation. Below, you see such V-patterns, each with a ranking number.

Find an expression of the number of dots N in terms of the ranking number R.

< multi-line formula editing window or open answer window>

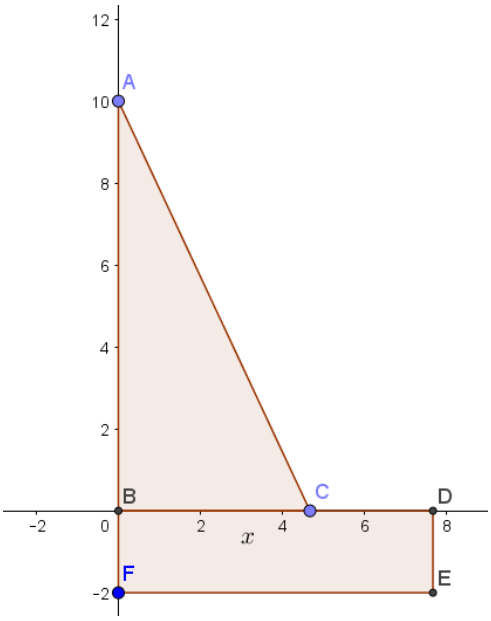
Possible steps:

Code	Solution step	Node user model
Ans1Strat1Step1	Recognize: 1 dot in center and R dots on each side	
Ans1Strat1Step2	N = 2*R + 1	R1213: Set up linear model
Ans1Strat2Step1	Recognize: R dots on one side and R+1 dots on the other side	
Ans1Strat2Step2	N = 2*R + 1	R1213: Set up linear model
Ans1Strat3Step1	Recognize: figure 1 has 3 dots	
Ans1Strat3Step2	Recognize: in step R, 2*(R-1) dots are added	
Ans1Strat3Step3	N = 3 + 2*(R–1)	R1213: Set up linear model
Ans1Strat3Step4	N = 2*R + 1	R3221: Expand expressions
Ans1Strat4Step1	Calculate starting value: 3 – 2 = 1	

Ans1Strat4Step2	Recognize: each step 2 dots are added	
Ans1Strat4Step3	N = 2*R + 1	R1213: Set up linear model

8. Area of a triangle (R1, R2 & R3)

Task (source Pépite)



In this applet, you can move point C along the horizontal axis.

Triangle ABC is a rectangular triangle. The length of AB is 10. The length of BC is x.

BDEF is a rectangle. The length of BF is 2. The length of CD is 3.

Q1 Express the area of ABC in terms of x.

< multi-line formula editing window or open answer window>

Q2 Express the area of BDEF in terms of x.

< multi-line formula editing window or open answer window>

Q3 Find the value of x for which the two areas are equal.

< multi-line formula editing window or open answer window>

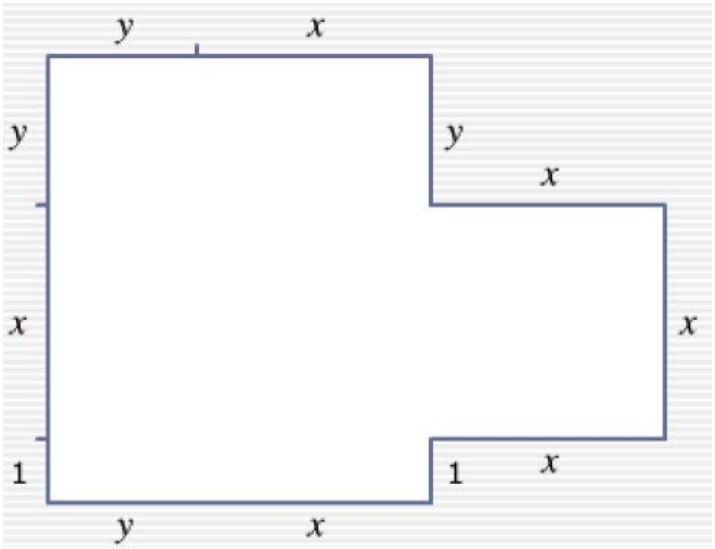
Possible steps:

Code	Solution step	Node user model
Ans1Strat1Step1	10 * x / 2	R1212: Set up geometric model R1213: Set up linear model
Ans1Strat1Step2	5*x	R3222: Divide out common factors
Ans2Strat1Step1	2 * (x + 3)	R1212: Set up geometric model R1213: Set up linear model
Ans2Strat1Step2	2*x + 6	R3221: Expand expressions
Ans3Strat1Step1	5*x = 2*x +6	R12: Set up algebraic equations

Ans3Strat1Step2	3*x = 6	R323: Simplify by addition and subtraction
Ans3Strat1Step3	x = 6 / 3 = 2	R324: Simplify by multiplication and division
Ans3Strat2Step1	Try <number> 5 * <number> ≠ 2 * <number> + 6	R321: Substitute
Ans3Strat2Step2	Try <2> 5 * 2 = 2 * 2 + 6	R321: Substitute
Ans3Strat2Step3	x = 2	

9. Area and expression (R2)

Task (source Pépite)



Please click on parts of the figure so that the area in total will be $x(x+y+1)+x^2$.

Possible steps:

Code	Solution step	Node user model
Ans1Strat1Step11	Answer: $\begin{matrix}0&1&0\\0&1&1\\0&1&0\end{matrix}$	R1212: Set up geometric model
Ans1Strat1Step12	Answer: $\begin{matrix}0&0&0\\1&1&1\\0&1&0\end{matrix}$	R1212: Set up geometric model

10. Matryoshka doll (R1 & R3)

Task (source CITO)

On your right hand side you see the first three of a series Matryoshka dolls. The puppets fit into each other, due to a decreasing height. The biggest puppet is 32 cm high. Each next puppet is 25% smaller than the previous one. In this sequence, there are no puppets smaller than 9 cm.

How many puppets are there in this series?
Write down your intermediate steps.

< multi-line formula editing window or open answer window>

Possible steps:

Code	Solution step	Node user model
Ans1Strat1Step1	Choose variable: h (height of puppet) Choose variable: n (number of puppet)	R1211: Choose variables
Ans1Strat1Step2	h = 32 * 0.75^(n-1)	R1215: Set up exponential model
Ans1Strat1Step3	Try n ≠ 6	R321: Substitute
Ans1Strat1Step4	Try n = 6 : 32 * 0.75^5 = 7.594	R321: Substitute
Ans1Strat1Step5	Answer: 5 puppets	
Ans1Strat2Step1	Second puppet: 32 * 0.75 = 24	R11: Set up numerical expressions R31: Use numerical expressions
Ans1Strat2Step2	Third puppet: 24 * 0.75 = 18	R11: Set up numerical expressions R31: Use numerical expressions
Ans1Strat2Step3	Fourth puppet: 18 * 0.75 = 13.5	R11: Set up numerical expressions R31: Use numerical expressions
Ans1Strat2Step4	Fifth puppet: 13.5 * 0.75 = 10.125	R11: Set up numerical expressions R31: Use numerical expressions
Ans1Strat2Step5	Sixth puppet: 10.125 * 0.75 = 7.594	R11: Set up numerical expressions R31: Use numerical expressions
Ans1Strat2Step6	Answer: 5 puppets	
Ans1Strat3Step1	First reduction: 32 * 0.25 = 8	R11: Set up numerical expressions R31: Use numerical expressions
Ans1Strat3Step2	Second puppet: 32 – 8 = 24	R11: Set up numerical expressions R31: Use numerical expressions
Ans1Strat3Step3	Second reduction: 24 * 0.25 = 6	R11: Set up numerical expressions R31: Use numerical expressions
Ans1Strat3Step4	Third puppet: 24 – 6 = 18	R11: Set up numerical expressions R31: Use numerical expressions
Ans1Strat3Step5	Third reduction: 18 * 0.25 = 4.5	R11: Set up numerical expressions R31: Use numerical expressions
Ans1Strat3Step6	Fourth puppet: 18 – 4.5 = 13.5	R11: Set up numerical expressions R31: Use numerical expressions
Ans1Strat3Step7	Fourth reduction: 13.5 * 0.25 = 3.375	R11: Set up numerical expressions

		R31: Use numerical expressions
Ans1Strat3Step8	Fifth puppet: 13.5 – 3.375 = 10.125	R11: Set up numerical expressions R31: Use numerical expressions
Ans1Strat3Step9	Fifth reduction: 10.125 * 0.25 = 2.531	R11: Set up numerical expressions R31: Use numerical expressions
Ans1Strat3Step10	Sixth puppet: 10.125 – 2.531 = 7.594	R11: Set up numerical expressions R31: Use numerical expressions
Ans1Strat3Step11	Answer: 5 puppets	

Mistakes recognized

M1: Mistakes with parentheses

In rectangle area:	Ans1Strat1Step1
	a + 5 * a + b
	a + 5 (a + b)
	(a + 5) a + b
In magic trick:	Ans1Strat1Step2
	(x+8)*3-4+x/4 + 2 – x
	(x+8*3-4+x)/4 + 2 – x
	x+8*3-4+x/4 + 2 – x
In magic trick:	Ans1Strat3Step1
	(<number> + 8*3 - 4 + <number>)/4 + 2 – <number>
	(<number> + 8)*3 - 4 + <number>/4 + 2 – <number>
	<number> + 8*3 - 4 + <number>/4 + 2 – <number>
In area of a triangle:	Ans2Strat1Step1
	2 * x + 3

M2a: Mix up multiplication and addition

M2b: Mix up division and subtraction

M3a: Mix up area and perimeter

In rectangle area:	Ans1Strat1Step1
	2*(a+5) + 2*(a+b)
In rectangle area:	Ans2Strat1Step2
	4a + 2ab + 2a + 10 + 2b + 10 =
	2ab + 6a + 2b + 20
In making a square:	Ans1Strat1Step3

4*(b-a)

In making a square: Ans1Strat1Step4

4*(sqrt (a^2+b^2))

In area of a triangle: Ans1Strat1Step1

10 + x + sqrt(10^2 + x^2)

In area of a triangle: Ans2Strat1Step1

2 * (x + 3 + 2)

M3b: Mix up area and volume

Any formula with ^3

M4: Mix up exponential model and linear model

In matryoshka: Any answer with heights: 32, 24, 16, 10.

In matryoshka: Ans1Strat1Step2

h = 32 + 0.75*n

M5: Combining unlike terms

Any mistake like 3x + 24 = 27x

M6: Incorrect management of priority of operations

Any mistake like (x + 8)*3 - 4 + x = (x+8)*-1 + x

A2 Competence in the domains of Number and Relationship

The list below is a domain model of competences needed by 15-year old students in the mathematical domains of Number and Relationships. In the Core Subject Taxonomy for Mathematical Sciences Education[1], these domains involve categories 1.2 Arithmetic and 3.1 Algebra.

The model is based on the intermediate achievement goals set up in the Netherlands[2], and adapted from the perspectives of similar descriptions from France and Germany[3], and the project's focus on tasks that invite multiple-step solutions. Also, the model reflects core research publications on algebra education, such as Kieran (2007), Grugeon-Allys et al. (2012) and Arcavi et al. (2017). As such, these competences form the point of departure for the selection of tasks in the Advise-Me project (task A1).

The Number domain

The student can …

N1. recognize and **set up arithmetic expressions** to solve problem situations;
N2. **carry out calculations** with small and large numbers, positive and negative numbers, powers and square roots, fractions and decimal numbers, making use of properties of numbers and operations;
N3. recognize and **solve tasks on ratio**, through the structuring of data and the use of the relationships between ratio, fraction, decimal number and percentage.

The Relationship domain

The student can …

R1. extend regularity, and **set up expressions, equations, and inequalities** for different problem situations, such as linear, quadratic or exponential expressions;
R2. produce or **relate different mathematical representations**, such as tables, graphs, formulas, and natural language;
R3. **simplify, expand, and factor algebraic expressions and solve equations and inequalities** (linear, quadratic, exponential), and substitute values for variables for further calculation.

References

Arcavi, A., Drijvers, P., & Stacey, K. (2017). *The Teaching and Learning of Algebra: Ideas, Insights and Activities.* London / New York: Routledge.

[1] Version 4/29/2005, see
[2] See http://www.slo.nl/downloads/documenten/tussendoelen-wiskunde-havo-vwo-onderbouw-vo.pdf for the original version in Dutch.
[3] E.g., see http://eduscol.education.fr/ressources-math and https://www.kmk.org/fileadmin/Dateien/veroeffentlichungen_beschluesse/2010/2010_04_29-Kerncurriculum-Mathematik.pdf and https://www.ac-strasbourg.fr/fileadmin/pedagogie/mathematiques/College/Programmes_Documents_officiels/Maths_cycle4_BO_SPE_11_26-11-2015.pdf .

Grugeon-Allys, B., Pilet, J., Chenevotot, F., & Delozanne, E. (2012). Diagnostic et parcours différenciés d'enseignement en algèbre élémentaire. *Recherches en Didactique des Mathématiques*, Enseignement de l'algèbre, bilan et perspectives, hors-série, 137-162.

Kieran, C. (2007). Learning and teaching algebra at the middle school through college levels: Building meaning for symbols and their manipulation. In Frank K. Lester (Eds.) *Second Handbook of Research on Mathematics Teaching and Learning* (pp.707–762). Greenwich, CT: Information Age Publishing.